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Some properties of size - biased weighted Weibull distribution

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ABSTRACT

This paper introduces a new distribution based on the Weibull distribution, known as Size biased weighted Weibull distribution (SWWD). Some characteristics of the new distribution are obtained. Plots for the cumulative distribution function, probability density function (pdf) and hazard function, tables with values of skewness and kurtosis are provided. We also provide results of entropies and characterization of SWWD. As a motivation, the statistical applications of the results to the problems of ball bearing data and snow fall data set have been provided. It is found that our recently proposed distribution fits better than size biased Rayleigh and Maxwell distributions.

1. Introduction

Weighted distributions are suitable in the situation of unequal probability sampling, such as actuarial sciences, ecology, biomedicine, biostatistics and survival data analysis. These distributions are applicable when observations are recorded without any experiment, repetition and random process. For more detail of weighted distribution see Zahida (2014-2015).

Let $f(x;\theta)$ be the pdf of the random variable x and θ be the the unknown parameter.

The weighted distribution is defined as;

$$g(x;\theta) = \frac{W(x)f(x;\theta)}{E[W(x)]} \text{ where } x \in R, \theta > 0. \tag{1}$$

Where w(x) is a weight function, defined as w(x) $=x^m$, where m = 1, it is called size biased weighted distribution.

1.1. Weibull distribution

Weibull Distribution is an important and well known distribution which attracted statisticians, working in various fields of applied statistics as well as theory and methods in modern statistic due to its number of special features and ability to fit to data related to various fields like as life testing, biology, ecology, economics, hydrology, engineering and business administration. This distribution is one of the members of the family of extreme value distributions. For more detail see Zahida (2015).

Provost et al. (2011) introduced some properties of three parameter weighted Weibull distribution. He defined the probability density function as

$$f(x;k,\xi,\theta) = \frac{k\,\theta^{\frac{\xi}{k}+1}\, x^{\xi+k-1}\, e^{-\theta x^k}}{\Gamma\left(1+\frac{\xi}{k}\right)}, \quad x > 0 \tag{2}$$

with $\xi + k > 0$, and with shape parameter ξ.

1.2. Size-biased weighted Weibull distribution

Theorem 1: Let X be a non- negative random variable, then the following relationship between Eq. 1 and the weight function w(x) can be defined as $g(x;\xi,\theta,k) = \frac{w(x)f(x)}{\int_0^\infty x\, f(x)dx}$ where w (x) is the first weight, w(x) = x and f(x) is defined in Eq. 2.

Proof: Suppose X has a density function g(x; ξ,k, θ) with unknown parameters ξ, θ, k. Using Eq.1 and Eq.2 , the corresponding distribution, named as weighted Weibull distribution is of the type:

$$g(x;\xi,\theta,k) = \frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}\, x^{\xi+k}\, e^{-\theta x^k}}{\Gamma(1+\frac{1+\xi}{k})}, \quad \xi,\theta,k, > 0, \; 0 < x < \infty \tag{3}$$

where ξ and k are shape parameters and θ is the scale parameter. The graphical representation of SWWD for different values of parameters is represented by Fig. 1.

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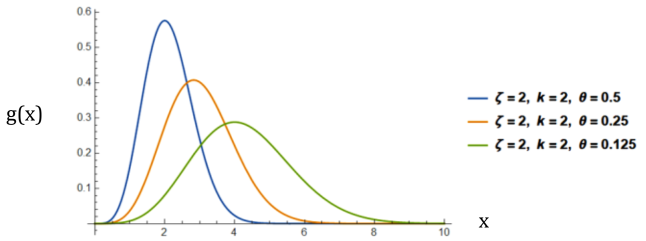

Fig. 1: Probability density function of SWWD for the indicated values of ξ, k and θ

Theorem 2: Let X be a non-negative random variable with the probability density function Eq. 3; then the cumulative distribution function (cdf) is defined as:

$$G(x,\xi,\theta,k) == 1-\frac{1}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\,\Gamma\left(1+\frac{1+\xi}{k},\theta x^{k}\right),\xi,k,\theta>0 \quad (4)$$

where $\Gamma(a,\ x) = \int_x^{\infty} y^{a-1}e^{-y}dy$ represents an incomplete gamma function.

Proof: Trivially using integration by part we have the requested form Eq. 4. Graph of distribution function is represented by Fig. 2.

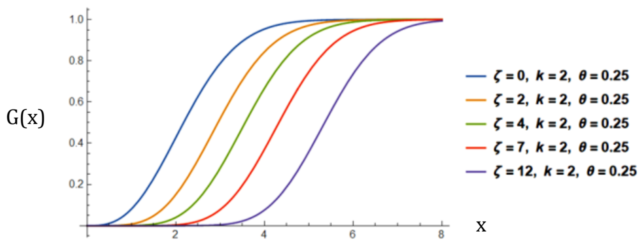

Fig. 2: Distribution function of SWWD for the indicated values of θ, ξ and k

1.3. The survival function of SWWD

The survival function of SWWD is defined as S(x) = 1 – G(x),

$$S(x)=\frac{\Gamma\left(1+\frac{1+\xi}{k},\ \theta x^{k}\right)}{\Gamma\left(1+\frac{1+\xi}{k}\right)},\quad \xi,k,\theta>0 \quad (5)$$

which is represented by Fig. 3.

1.4. The hazard rate function of SWWD

The hazard function is the instant level of failure at a certain time. Characteristics of a hazard function are normally related with definite products and applications. Different hazard functions are displayed with different distribution models. Some properties of Hazard rate were pointed out by Nadarajah and Kotz (2004). The reliability measures of weighted distributions were evaluated by Dara and Ahmad (2012).

Theorem 3: Suppose X be a non-negative random variable with the probability density function then hazard rate is defined by $h(x) = \frac{g(x)}{S(x)}$, where S(x) is the survival function (Fig. 4).

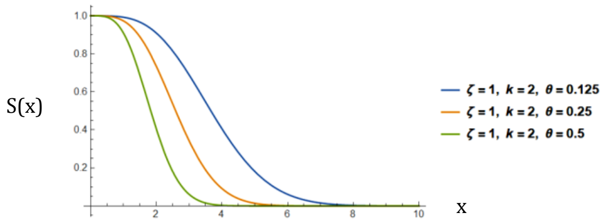

Fig. 3: Survival function of SWWD for the indicated values of θ, ξ and k

Proof: By putting values of pdf and S(x) in above formula, at c =1 and $\boldsymbol{\lambda}$ **=** 1, hazard rate will be

$$h(x)=\frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}\,x^{\xi+k}\,e^{-\theta x^{k}}}{\Gamma\left(1+\frac{1+\xi}{k},\ \theta x^{k}\right)},\quad \xi,k,\theta>0 \quad (6)$$

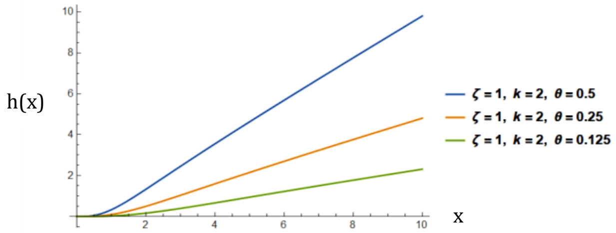

Fig. 4: Hazard rate function of SWWD for the indicated values of ξ, k and θ

Corollary 1: Let X be a non - negative random variable then Reverse hazard function is the quotient of pdf and hazard rate e.g., $r(x) = \frac{g(x)}{G(x)}$

Proof: By putting required values in above formula, we have the following,

$$r(x)=\frac{g(x)}{G(x)}=\frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}\,x^{\xi+k}\,e^{-\theta x^{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)-\Gamma\left(1+\frac{1+\xi}{k},\ \theta x^{k}\right)}\quad \theta,k,\xi>0 \quad (7)$$

The graph of reverse hazard function is given by Fig. 5.

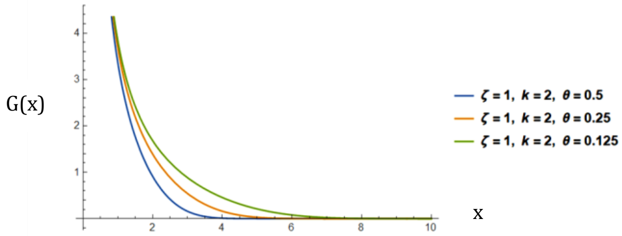

Fig. 5: Reverse hazard rate function of SWWD for the indicated values of ξ, k and θ

2. Moments (r^{th} moments about zero)

Suppose X is a random variable with pdf g(x) as given in Eq. 3. Then using gamma function r^{th} moment is easily expressed as

The r^{th} moments about zero is

$$\mu_r^{/} = E(x^r) = a_r\, \theta^{\frac{-r}{k}} \quad (8)$$

where $a_r = \frac{\Gamma\left(1+\frac{1+r+\xi}{k}\right)}{\Gamma\left(1+\frac{1+\xi}{k}\right)}$, r= 1, 2, 3….

The four moments can be obtained by putting r=1,2,3,4 about the mean are:

$\mu_1 = 0$

$\mu_2 = a_2\theta^{\frac{-2}{k}} - a_1{}^2\theta^{\frac{-2}{k}}$

$= (a_2 - a_1{}^2)\theta^{\frac{-2}{k}}$

$\mu_3 = a_3\theta^{\frac{-3}{k}} - 3a_1a_2\theta^{\frac{-3}{k}} + 2a_1^3\theta^{\frac{-3}{k}}$

$= \theta^{\frac{-3}{k}}(a_3 - 3a_1a_2 + 2a_1^3)$

$\mu_4 = a_4\theta^{\frac{-4}{k}} - 4a_1a_3\theta^{\frac{-4}{k}} + 6a_1^2a_2\theta^{\frac{-4}{k}} - 3a_1^4\theta^{\frac{-4}{k}}$

$= \theta^{\frac{-4}{k}}\,(a_4 - 4a_1a_3 + 6a_1^2a_2 - 3a_1^4)$

where a_1, a_2, a_3 and a_4 are defined in Eq. 8. The measure of coefficient of skewness and kurtosis for SWWD is given by Table 1.

Table 1: The measure of coefficient of skewness and kurtosis for SWWD

k	2.9	3.0	3.1	3.2	3.3
θ	0.5	0.5	0.5	0.5	0.5
ξ	1.0	1.0	1.0	1.0	1.0
$\sqrt{\beta_1}$	0.084	0.079	0.055	0.032	0.010
β_2	2.823	2.824	2.825	2.824	2.825

It is clear from Table 1 that SWWD is almost symmetrical and platykurtic for $2.9 \le k \le 3.3$.

3. Limit of the function

The limits of the density function given in Eq. 3 are as follows:

$$\lim_{x\to 0} g(x;\, k, \theta,\, \xi) = 0 \quad (9)$$

$$\lim_{x\to 0} g(x;\, k, \theta,\, \xi) = 0 \quad (10)$$

4. Mode of SWWD

Mode of Eq. 3 can be found by solving equation $\frac{\partial}{\partial x}$ Log $(g(x;\, k, \theta,\, \xi) = 0$, we have mode $= \langle\frac{\xi+k}{\theta\, k}\rangle^{\frac{1}{k}}$

The mode of Eq. 3 for different values of parameters is given by the Tables 2, 3 and 4.

Table 2: Mode of SWWD for values $\xi = 1$, k = 2

θ	0.500	0.450	0.400	0.350	0.300	0.250
Mode	1.732	1.825	1.936	2.070	2.236	2.449

Table 3: Mode of SWWD for values θ = 0.5, ξ = 1

k	2.900	3.000	3.100	3.200	3.300	3.400
Mode	1.406	1.386	1.357	1.352	1.336	1.322

Table 4: Mode of SWWD for values θ = 0.5, k = 3

θ	0.500	0.450	0.400	0.350	0.300	0.250
Mode	1.326	1.386	1.442	1.493	1.542	1.587

5. Moment generating function (mgf)

The mgf of SWWD is given as:

$$M_X(t) = \int_0^\infty e^{tx} g(x)\, dx$$

Using value of g(x) from Eq. 3 and after some simplification:

$$M_X(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \frac{\theta^{\frac{-i}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\, \Gamma\left(1+\frac{1+i+\xi}{k}\right) \quad (11)$$

6. Estimation of parameters

Maximum likelihood (ML) Estimation is used to estimate the parameters of SWWD. If $X_1, X_2 \dots \dots X_n$ be a random sample from a population having pdf $g(x|k, \theta, \xi)$, the likelihood function of SWWD distribution may be defined as:

$$L\,(\theta, \xi, k; x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} g(x_i).$$

$x_1, x_2, \dots, x_n$ are the independent observations, then the log likelihood function of the distribution is:

$$L\,(\theta, \xi, k; x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \log\,(g(\,x_i;\, \theta, \xi, k) = n \log k + n\,(1+\frac{1}{k}+\frac{\xi}{k}) \log\theta$$

$$-n\log\Gamma\left(1+\frac{1+\xi}{k}\right) + (k+\xi)\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^k \quad (12)$$

ML estimates can be found by solving Equations

$$\frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial \xi} = 0, \quad \frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial \theta} = 0, \quad \frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial k} = 0$$

$$\frac{n}{\hat{k}}\log(\hat{\theta}) + \sum_{i=1}^{n}\log(x_i) - \frac{n}{\hat{k}}\Psi^{(0)}\left(1+\frac{1+\hat{\xi}}{\hat{k}}\right) = 0, \quad (13)$$

where $\Psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

$$\frac{n\left(\hat{k} - (1+\hat{\xi})\log(\hat{\theta}) + (1+\hat{\xi})\Psi^{(0)}\left(1+\frac{1+\hat{\xi}}{\hat{k}}\right)\right)}{\hat{k}^2} + \sum_{i=1}^{n}\log(x_i) -$$

$$\hat{\theta}\sum_{i=1}^{n} x_i^{\hat{k}}\log(x_i) = 0 \quad (14)$$

$$n\left(1+\frac{1}{\hat{k}}+\frac{\hat{\xi}}{\hat{k}}\right)\cdot\frac{1}{\hat{\theta}} - \sum_{i=1}^{n} x_i^{\hat{k}} = 0 \quad (15)$$

Eqs. 13, 14, and 15 are nonlinear equations and can be solved through Mathematica software.

Asymptotic variance-covariance matrix is the inverse of $I(\theta, k, \xi) = -E(H(X))$

$$H(x) = \begin{pmatrix} \frac{\partial^2(\log(g(X;\theta,k,\xi)}{\partial\xi^2} & \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\xi\,\partial k)} & \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\xi\,\partial\theta)} \\ \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial k\,\partial\xi)} & \frac{\partial^2(\log(g(X;\theta,k,\xi))}{\partial k^2} & \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial k\,\partial\theta)} \\ \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\theta\,\partial\xi)} & \frac{\partial^2(\log(g(X;\theta,k,\xi)}{(\partial\theta\,\partial k)} & \frac{\partial^2(\log(g(X;\theta,k,\xi))}{\partial\theta^2} \end{pmatrix} \quad (16)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi))}{\partial\xi^2} = \frac{-n\Psi^{(1)}\left(1+\frac{1+\xi}{k}\right)}{k^2} \quad (17)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi))}{\partial k^2} = \frac{-n}{k^2} + \frac{2n(1+\xi)}{k^3}\log(\theta) - \theta\sum_{i=1}^{n} x_i^k \log(x_i)^2$$

$$-\frac{2n}{k^3}(1+\xi)\Psi^{(0)}\left(1+\frac{1+\xi}{k}\right) - \frac{n(1+\xi)^2}{k^4}\Psi^{(1)}\left(1+\frac{1+\xi}{k}\right) \quad (18)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi))}{\partial\theta^2} = \frac{-n}{\theta^2}\left(1+\frac{1}{k}+\frac{\xi}{k}\right) \quad (19)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\theta\,\partial k)} = \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial k\,\partial\theta)} = \frac{n}{\theta}\left(\frac{-1}{k^2} - \frac{\xi}{k^2}\right) - \sum_{i=1}^{n} x_i^k \log(x_i) \quad (20)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\xi\,\partial\theta)} = \frac{\partial^2(\log(g(X;\theta,k,\xi))}{(\partial\theta\,\partial\xi)} = \frac{n}{\theta k} \quad (21)$$

$$\frac{\partial^2(\log(g(X;\theta,k,\xi)))}{(\partial\xi\,\partial k)} = \frac{\partial^2(\log(g(X;\theta,k,\xi)))}{(\partial k\,\partial\xi)} = \frac{-n}{k^2}\log(\theta) + \frac{n}{k^2}\Psi^{(0)}\left(1+\frac{1+\xi}{k}\right) + \frac{n}{k^3}(1+\xi)\Psi^{(1)}\left(1+\frac{1+\xi}{k}\right) \quad (22)$$

7. Recurrence relation of size biased weighted Weibull distribution

Theorem 4: Let x be the random variable on its support $(0,\infty)$. Then recurrence relation through conditional moments for all $t>0$

$$E\,(X^{nk}|X>t) = \frac{\theta^{n+\frac{1+\xi}{k}} t^{nk+(1+\xi)} e^{-\theta t^k}}{\Gamma(1+\frac{1+\xi}{k},\ \theta t^k)\theta^n} + \frac{\left(n+\frac{1+\xi}{k}\right)}{\theta} E(X^{(n-1)k}|X>t) \quad (23)$$

where $\xi, \theta, k > 0$ and $n \in z^+$

Proof: Let X be the size biased weighted Weibull distribution. Then

$$E\,(X^{nk}|X>t) = \frac{1}{\bar{G}_1(t)}\int_t^\infty x^{nk} g(x)\,dx$$

using

$$\bar{G}(t) = \frac{\Gamma\left(1+\frac{1+\xi}{k},\ \theta t^k\right)}{\Gamma(1+\frac{1+\xi}{k})}$$

$$= \frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k},\ \theta t^k\right)}\int_t^\infty x^{nk+\xi+k} e^{-\theta x^k} dx$$

using the transformation $\theta x^k = u$, $k\,\theta\, x^{k-1}dx = du$ and $\theta t^k < u < \infty$

$$= \int_{\theta t^k}^\infty \left(\frac{u}{\theta}\right)^n \left(\frac{u}{\theta}\right)^{\frac{\xi+k}{k}} e^{-u} . \frac{du}{\theta\left(\frac{u}{\theta}\right)^{\frac{k-1}{k}}}$$

$$= \frac{\theta^{\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k},\theta t^k\right)\theta^n}\int_{\theta t^k}^\infty (u)^{n+\frac{1}{k}+\frac{\xi}{k}} e^{-u} du \quad (24)$$

Integration by parts, we get:

$$= \frac{\theta^{n+\frac{1+\xi}{k}} t^{nk+(1+\varepsilon)} e^{-\theta t^k}}{\Gamma\left(1+\frac{1+\xi}{k},\theta t^k\right)\theta^n} + \frac{\left(n+\frac{1}{k}+\frac{\xi}{k}\right)}{\theta\Gamma\left(1+\frac{1+\xi}{k},\theta t^k\right)\theta^{n-1}}\int_{\theta t^k}^\infty u^{(n-1)+\frac{1+\xi}{k}} e^{-u} du \quad (25)$$

After some simplification, we will obtain Eq. 23

Corollary. If $\xi = 0$ then Eq. 25 reduces for 2-parameters Weibull distribution.

8. Entropy

Entropy is considered as a major tool in every field of science and technology. In Statistics entropy is considered as an amount of incredibility. Different ideas of entropy have been given by Jaynes (1980) and the entropies of continuous probability distributions have been approximated by Ma (1981). Shanon entropy is defined as h(X) of a continuous random variable X with a density function f(x) (Jeffrey and Zwillinger, 2007)

$$h(X) = E\,[-\log(f(x))]$$

$$h\,[g\,(x;\,\theta,\xi,k)] = E[-\log g(x;\ \theta,\xi,k)]$$
$$= E\left[-\log\left\{\frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}} x^{\xi+k} e^{-\theta x^k}}{\Gamma(1+\frac{1+\xi}{k})}\right\}\right] = E\left[\theta\, x^k - \log(k) - (k+\xi)\log(x) - \left(1+\frac{1}{k}+\frac{\xi}{k}\right)\log(\theta) + \log\Gamma\left(1+\frac{1+\xi}{k}\right)\right] = E(\theta\, x^k) - \log(k) - (k+\xi)E\log(x) - \left(1+\frac{1}{k}+\frac{\xi}{k}\right)\log(\theta) + \log\Gamma\left(1+\frac{1+\xi}{k}\right) = \log\frac{\Gamma\left(1+\frac{1+\xi}{k}\right)}{k} - \left(1+\frac{1}{k}+\frac{\xi}{k}\right)\log(\theta) - (\xi+k)\,E\log(x) + E[\theta\, x^k] \quad (26)$$

$E\log(x)$

$$= \frac{1}{k}\frac{1}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\int_0^\infty \log\left(\frac{t}{\theta}\right).t^{\frac{1+\xi}{k}} e^{-t}dt = \frac{1}{k}\frac{1}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\left[\int_0^\infty \log t.t^{\frac{1+\xi}{k}}\ e^{-t}dt - \int_0^\infty \log\theta\ t^{\frac{1+\xi}{k}}\ e^{-t}dt\right] \int_0^\infty \log x\ x^{\gamma-1} e^{-x}\,dx = \Gamma'(\gamma)$$

$$E\log(x) = \frac{1}{k}\frac{1}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\left[\Gamma'\left(1+\frac{1+\xi}{k}\right) - \log\theta\Gamma\left(1+\frac{1+\xi}{k}\right)\right] \quad (27)$$

$$E[\theta x^k] = \frac{k\theta^{2+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\Gamma\left(2+\frac{1+\xi}{k}\right) \quad (28)$$

Putting Eq. 27 and Eq. 28 in Eq. 26

$$h[g(x;\ \xi,k,\theta)] = \log\frac{\Gamma\left(1+\frac{1+\xi}{k}\right)}{k} - \left(1+\frac{1}{k}+\frac{\xi}{k}\right)\log(\theta) - (\xi+k)\frac{1}{k}\frac{1}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\left(\Gamma'\left(1+\frac{1+\xi}{k}\right) - \log\theta\Gamma\left(1+\frac{1+\xi}{k}\right)\right) + \frac{k\,\theta^{2+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\Gamma\left(2+\frac{1+\xi}{k}\right)$$

where $\xi, k, \theta > 0$.

Renyi (1961) entropy is usually known as the generalized procedure of Shannon entropy. The Renyi entropy is named after. It is useful in ecology and statistics. It is defined as

$$I_R(\beta) = \frac{1}{1-\beta}\log(\int_0^\infty g^\beta(x)dx)\quad \beta>0, \beta\neq 1)$$

Putting value of g(x) from Eq. 3 in above equation, we get:

$$= \frac{1}{1-\beta}\log\left[\int_0^\infty\left[\frac{k\theta^{1+\frac{\xi}{k}+\frac{1}{k}}x^{\xi+k}e^{-\theta x^k}}{\Gamma(1+\frac{1+\xi}{k})}\right]^\beta\right]$$

take

$$g^\beta(x;\theta,\xi,k) = \left[\frac{k\theta^{1+\frac{\xi}{k}+\frac{1}{k}}x^{\xi+k}\,e^{-\theta x^k}}{\Gamma\left(1+\frac{1+\xi}{k}\right)}\right]^\beta$$

$$\int_0^\infty g^\beta(x;\theta,\xi,k)dx = \frac{k^\beta\theta^{\beta+\frac{\beta}{k}+\frac{\beta\xi}{k}}}{\left[\Gamma\left(1+\frac{1+\xi}{k}\right)\right]^\beta}\int_0^\infty x^{k\beta+\beta\xi}\ e^{-\beta\,\theta x^k}dx$$

$$I_R(\beta) = \frac{\beta-1}{k}\log\theta + \log\Gamma\left(\beta+\frac{1+\beta\xi}{k}\right) - \log k - \left(\beta+\frac{1}{k}+\frac{\beta\xi}{k}\right)\log\beta - \beta\log\Gamma\left(1+\frac{1+\xi}{k}\right).$$

9. Characterization of size biased weighted Weibull distribution

A characterization is a definite distributional property of statistics that uniquely defines the related stochastic model. There are some functions related to a probability distribution that uniquely classify it. Such functions are called characterizing functions. Here we are characterizing the SWWD distribution through conditional moments by using the characterizing function $f(x)$.

Theorem 5: Let X be the random variable on its support $(0, \infty)$. Then SWWD can be characterized as

$$E\left[(u(x)|X>t)\right]=\frac{pe^{-\theta t^k}}{\theta k\bar{G}(t)},\ \theta,k>0,$$

where

u(x) = $x^{-\varepsilon-1}$

and p is constant.

Proof:

$$E\left[(g(x)|X>t)\right]=\frac{1}{\bar{G}(t)}\int_t^{\infty}x^{-1-\xi}g(x)dx$$

$$=\frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)\bar{G}(t)}\int_t^{\infty}x^{-1+k}\,e^{-\theta x^k}dx$$

$$=\frac{p}{\theta k\bar{G}(t)}\int_t^{\infty}\theta\,k\,x^{-1+k}\,e^{-\theta x^k}dx$$

using the transformation, $\theta x^k=u$, we have:

$$=\frac{p}{\theta k\bar{G}(t)}\int_{\theta t^k}^{\infty}e^{-u}du$$

$$=\frac{p}{\theta k\bar{G}(t)}e^{-\theta t^k}$$

conversely

$$\frac{1}{\bar{G}(t)}\int_t^{\infty}x^{-1-\varepsilon}g(x)dx=\frac{P}{\theta k\bar{G}(t)}e^{-\theta t^k}$$

differentiating both sides w.r.t, 't', we get:

$$-t^{-1-\varepsilon}g(t)=\frac{Pe^{-\theta t^k}}{\theta k}\left(-\theta k t^{k-1}\right)$$

after simplification:

$$g\,(t)=pe^{-\theta t^k}t^{\varepsilon+k}\qquad \theta,\mathrm{k},\xi>0$$

where $P=\frac{k\theta^{1+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)}$ is constant.

10. Numerical examples

10.1. The ball bearing data records

See for data set published in Lawless (p.228, 19.82). Table 5 shows the goodness-of-fit statistics and parameters' estimates.

Table 5: The ball bearing data records

Distributions	$\hat{\theta}$	$\hat{k}$	$\hat{\xi}$	$\hat{\alpha}$	$\hat{\beta}$	A_0^2	W_0^2
Size Biased Rayleigh	-	-	-	-	46.764	0.708	0.134
Size Biased Maxwell	-	-	-	40.50	-	1.693	0.278
Weighted Weibull (Size Biased)	0.8151	0.6047	4.7599	-	-	0.1909	0.0332

In Table 5, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, the weighted Weibull model proposals the best fitting.

The comparison of Rayleigh (Dotted Dashed Line), Maxwell (Solid Line) and Size Biased Weighted Weibull (Short Dashes) on the Histogram is represented in Fig. 6.

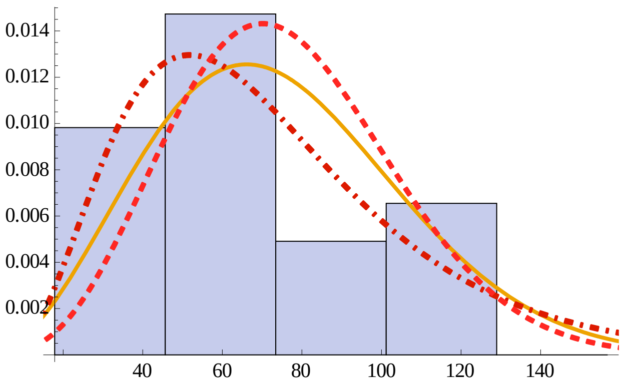

Fig. 6: The comparison of Rayleigh (Dotted Dashed Line), Maxwell (Solid Line) and Size Biased Weighted Weibull (Short Dashes)

The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf are represented in Fig. 7.

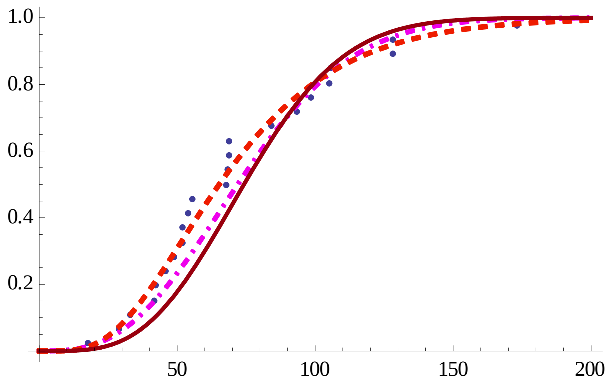

Fig. 7: The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf

10.2. The buffalo snowfall data set

See for data set Silverman (1986). Table 6 shows the goodness-of-fit statistics and parameters' estimates.

In Table 6, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, the weighted Weibull model proposals the best fitting.

The comparison of Rayleigh (Spotted Line), Two Parameter Weibull (Solid Line), Weighted Weibull (size biased) Distribution (Dashed Line) and Maxwell (Dotted Dashed) on the histogram is represented in Fig. 8.

The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf for the Snowfall Data are represented in Fig. 9.

Table 6: The buffalo snowfall data set

Distributions	$\hat{\theta}$	$\hat{k}$	$\hat{\xi}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	A_0^2	W_0^2
Two parameter Weibull	3.37477 $\times 10^{-8}$	3.8338	-	88.898	-	-	0.3063	0.0454
Size biased Rayleigh	-	-	-	-	-	48.3095	2.385	0.4053
Size biased Maxwell	-	-	-	-	41.8373	-	0.9932	0.1647
Weighted Weibull (size biased)	0.0001038	2.2616	2.0972				0.3049	0.0454

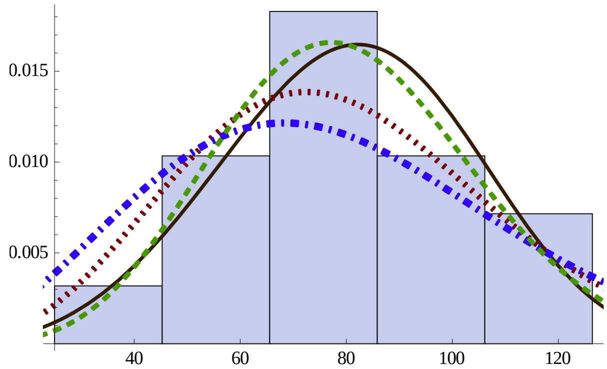

Fig. 8: The comparison of Rayleigh (Spotted Line), Two Parameter Weibull (Solid Line), Weighted Weibull (size biased) Distribution (Dashed Line) and Maxwell (Dotted Dashed)

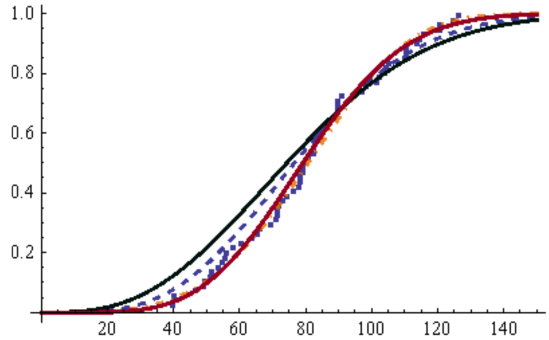

Fig. 9: The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf for the Snowfall Data

10.3. The strength data set

See for data set Bader and Priest (1982). Table 7 shows the goodness-of-fit statistics and parameters' estimates.

In Table 7, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, the weighted Weibull model proposals the best fitting.

Table 7: The strength data set

Distributions	$\hat{\xi}$	$\hat{\theta}$	$\hat{k}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	A_0^2	W_0^2
Two Parameter Weibull	-	0.0023	5.0494	3.3147		-	0.9325	0.1241
Size biased Rayleigh	-	-	-	-		1.8017	7.2704	1.3262
Size biased Maxwell	-	-	-	-	1.56035	-	5.0072	0.8414
Weighted Weibull (size biased)	21.5185	6.5258	1.0793	-	-	-	0.3727	0.0610

The comparison of Rayleigh (Dotted Line), Weighted Weibull Distribution (Solid), Maxwell (Dotted Dashed) and Two Parameter Weibull (Dashed Line) on the Histogram for the strength data is represented by the Fig. 10.

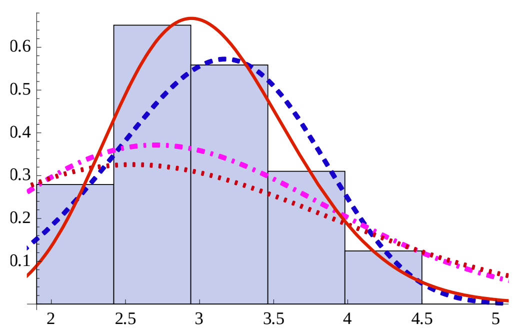

Fig. 10: The comparison of Rayleigh (Dotted Line), Weighted Weibull Distribution (Solid), Maxwell (Dotted Dashed) and Two Parameter Weibull (Dashed Line) on the Histogram for the strength data

The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf for the Strength Data are represented in Fig. 11.

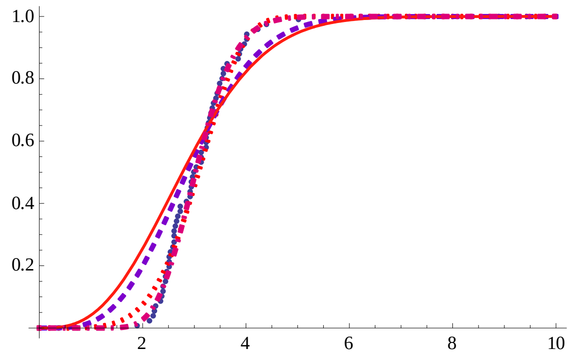

Fig. 11: The cumulative distribution function estimates, Size Biased Weighted Weibull Density estimates and Empirical cdf for the Strength Data

11. Conclusion

In this paper, we discussed the Size Biased Weighted Weibull Distribution (SWWD). The pdf of the SWWD has been obtained as well as different reliability measures. The moments, mode, the coefficient of skewness and the coefficient of kurtosis of SWWD have been derived. We also provide results of entropies and characterization of SWWD. For estimating the parameters of SWWD, MLE method has been used. The SWWD have been fitted to two kinds of data sets. SWWD suggested a good fit of the data as comparing to other distributions.

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